

Functionals on Closed 2-Surfaces

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Abstract

We show that the 2-torus in \mathbb{R}^3 is a critical point of a sequence of functionals \mathcal{F}_n ($n = 1, 2, 3, \dots$) defined over compact 2-surfaces in \mathbb{R}^3 . When the Lagrange function \mathcal{E} is a polynomial of degree n of the mean curvature H of the surface, the radii (a, r) of the 2-torus are related as $\frac{a^2}{r^2} = \frac{n^2-n}{n^2-n-1}$, $n \geq 2$. If the Lagrange function depends on both mean and Gaussian curvatures, the 2-torus remains to be a critical point of \mathcal{F}_n without any constraints on the radii of the torus.

1 Introduction

In the history of differential geometry there are some special subclasses of 2-surfaces, such as surfaces of constant Gaussian curvature, surfaces of constant mean curvature, minimal surfaces and the Willmore surfaces. These surfaces arise in many different branches of sciences; in particular, in various parts of theoretical physics (string theory, general theory of relativity), cell-biology and differential geometry [1]-[15]. All these special surfaces constitute critical points of certain functionals. Although the Euler-Lagrange equations of these functionals are very complicated and difficult, there are some techniques to solve these equations. Using the deformation of the Lax equations of the integrable equations it is possible to construct 2-surfaces in \mathbb{R}^3 solving more general shape equations [6]-[8].

The main objective in our work is to investigate 2-surfaces derivable from a variational principle, such as the minimal surfaces, Willmore surfaces [4], [5] and surfaces solving the shape equation [10]-[17]. In all these cases the Lagrange function is a quadratic function of the mean curvature of the surface. It is natural to ask whether there are surfaces solving the Euler-Lagrange equations corresponding to more general Lagrange functions depending on the mean and Gaussian curvatures of the surface [10]-[12]. It is the purpose of this work to find an answer to such a question.

The quadratic Helfrich functional [9] for a theoretical model of a closed cell-membrane is

$$\mathcal{F} = \frac{1}{2} \int_S [k_c (2H + c_0)^2 + 2w] dA + p \int_V dV \quad (1)$$

where k_c is the elasticity constant, H and c_0 are the mean and the spontaneous curvatures, w is the surface tension and p is the pressure difference between in and out of the surface. First variation of the above functional gives the shape equation

$$p - 2wH + 2k_c \nabla^2 H + k_c (2H + c_0)(2H^2 - c_0H - 2K) = 0 \quad (2)$$

Sphere with an arbitrary radius is an exact solution and shape equation (2) gives a relation between the model parameters and the radius of the sphere. Stability of this solution has been studied in [13]. A special 2-torus, called the Clifford torus, is also an exact solution of (2) [10]. We might consider that Helfrich's functional (1) as an approximation of the higher order functionals $\mathcal{F}_n = \int_S \mathcal{E}_n dA + p \int_V dV$, ($n \geq 2$) where

$$\begin{aligned} \mathcal{E}_n &= \sum_{k=0}^n a_{n+1-k} H^k, \quad n = 1, 2, \dots \\ &= a_1 H^n + a_2 H^{n-1} + \dots + a_n H + a_{n+1}, \end{aligned} \quad (3)$$

Here a_i 's are constants describing the parameters of the cell-membrane model. Hence it is worthwhile to study such functionals and search for possible critical points. This is another motivation of this work on functionals on closed surfaces.

In section 2 we introduce the the first and second variations of a functional $\mathcal{F} = \int_S \mathcal{E} dA + p \int_V dV$ where the Lagrange function is an arbitrary function

of the mean and Gauss curvatures of the surface S [10]. In section 3 we consider the Lagrange function \mathcal{E} depends only on the the mean curvature. Letting \mathcal{E}_n be a polynomial of degree n , then we show that 2-torus T_n is a critical point of the corresponding functional $\mathcal{F}_n = \int_S \mathcal{E}_n dA + p \int_V dV$ with $\frac{a^2}{r^2} = \frac{n^2-n}{n^2-n-1}$, $n = 2, 3, 4, \dots$. When $n = 2$ the corresponding torus T_2 is the Clifford torus. In section 4 we consider \mathcal{E}_n depend on both mean and Gaussian curvatures. We show that, inclusion of suitable Gaussian curvature terms into \mathcal{E}_n , 2-torus with arbitrary radii (a, r) becomes a critical point. In section 5 we prove two theorems: First one is to prove the constraint equation for the radii of the 2-torus and the second one is on the more general functionals without the constraint equation. In Appendix A and B we give more examples of functionals discussed in sections 3 and 4.

2 Surfaces from a variational principle

Let S be a 2-surface in \mathbb{R}^3 with Gaussian (K) and mean (H) curvatures. A functional \mathcal{F} is defined by

$$\mathcal{F} = \int_S \mathcal{E}(H, K) dA + p \int_V dV \quad (4)$$

where \mathcal{E} is the Lagrange function depending on H and K . Functional \mathcal{F} is also called curvature energy or shape energy. Here p is a constant which play the role of Lagrange multiplier and V is the volume enclosed within the surface S . We obtain the Euler-Lagrange equations corresponding to the above functional from its first variation. Let \mathcal{E} be a twice differentiable function of H and K . Then the first variation of \mathcal{F} is given by

$$\delta \mathcal{F} = \int_S E(\mathcal{E}) \Omega dA. \quad (5)$$

where Ω is an arbitrary smooth function on S . Then the Euler-Lagrange equation $E(\mathcal{E}) = 0$ for \mathcal{F} reduces to [10]-[12]

$$(\nabla^2 + 4H^2 - 2K) \frac{\partial \mathcal{E}}{\partial H} + 2(\nabla \cdot \bar{\nabla} + 2KH) \frac{\partial \mathcal{E}}{\partial K} - 4H\mathcal{E} + 2p = 0. \quad (6)$$

Here and in what follows, $\nabla^2 = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (\sqrt{g} g^{ij} \frac{\partial}{\partial x^j})$ and $\nabla \cdot \bar{\nabla} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (\sqrt{g} K h^{ij} \frac{\partial}{\partial x^j})$, $g = \det(g_{ij})$, g^{ij} and h^{ij} are inverse components of the first and second fundamental forms; $x^i = (u, v)$ and we assume Einstein's summation convention on repeated indices over their ranges.

For the second variation of the functional we assume that \mathcal{E} depends only on H . In this case the expression is much simpler [10]

$$\begin{aligned} \delta^2 \mathcal{F} = & \int_S \left(\mathcal{E}_1 \Omega^2 + \mathcal{E}_2 \Omega \nabla^2 \Omega - 2 \frac{\partial \mathcal{E}}{\partial H} \Omega \nabla \cdot \tilde{\nabla} \Omega + \frac{1}{4} \frac{\partial^2 \mathcal{E}}{\partial H^2} (\nabla^2 \Omega)^2 \right. \\ & \left. + \frac{\partial \mathcal{E}}{\partial H} [\nabla(H\Omega) \cdot \nabla \Omega - \nabla \Omega \cdot \tilde{\nabla} \Omega] \right) dA \end{aligned} \quad (7)$$

where

$$\begin{aligned} \mathcal{E}_1 &= (2H^2 - K)^2 \frac{\partial^2 \mathcal{E}}{\partial H^2} - 2HK \frac{\partial \mathcal{E}}{\partial H} + 2K\mathcal{E} - 2Hp, \\ \mathcal{E}_2 &= (2H^2 - K) \frac{\partial^2 \mathcal{E}}{\partial H^2} + 2H \frac{\partial \mathcal{E}}{\partial H} - \mathcal{E}, \end{aligned} \quad (8)$$

To have minimal energy solutions of (6) it is expected that the second variation $\delta^2 \mathcal{F} > 0$.

We have the following classical examples:

- i) Minimal surfaces: $\mathcal{E} = 1$, $p = 0$.
- ii) Constant mean curvature surfaces: $\mathcal{E} = 1$.
- iii) Linear Weingarten surfaces: $\mathcal{E} = aH + b$, where a and b are some constants.
- iv) Willmore surfaces: $\mathcal{E} = H^2$ [4], [5].
- v) Surfaces solving the shape equation (2) of lipid bilayer cell membranes:
 $\mathcal{E} = \frac{1}{2} k_c (2H + c_0)^2 + w$, where k_c , c_0 and w are constants [13]-[12].

2-sphere in \mathbb{R}^3 has constant mean and Gaussian curvatures. Hence it is a critical point of the most general functional (4). Eq. (6) gives a relation between p , radius of the sphere and other parameters in model.

Another compact surface in \mathbb{R}^3 is the 2-torus, T . It has been shown [17] that a special kind of torus, known as the Clifford torus, solves the shape equation (example (v) above). In this work we shall show that, T solves not only a quadratic shape equation but it solves all Euler-Lagrange equations with Lagrange function \mathcal{E} is a polynomial function of the mean and Gaussian curvatures. 2-Torus in \mathbb{R}^3 is defined as $X : U \rightarrow \mathbb{R}^3$ where [3]

$$X(u, v) = ((a + r \cos u) \cos v, (a + r \cos u) \sin v, r \sin u), \quad (9)$$

$$0 < u < 2\pi, 0 < v < 2\pi \quad (10)$$

The first and second fundamental forms are

$$ds_1^2 = g_{ij} dx^i dx^j = r^2 du^2 + (a + r \cos u)^2 dv^2, \quad (11)$$

$$ds_2^2 = h_{ij} dx^i dx^j = r^2 du^2 + (a + r \cos u) \cos u dv^2 \quad (12)$$

The Gaussian (K) and mean (H) curvatures are

$$K = \frac{\cos u}{r(a + r \cos u)}, \quad H = \frac{1}{2} \left(\frac{1}{r} + \frac{\cos u}{(a + r \cos u)} \right) \quad (13)$$

where a and r ($a > r$) are the radii of the torus. K and H satisfy a linear equation $r^2 K - 2rH + 1 = 0$. Hence torus is a linear Weingarten surface.

3 Functionals with Mean Curvature

In this section we shall consider the Lagrange function \mathcal{E} depending only on the mean curvature H of the surface. Furthermore we shall assume that \mathcal{E} is a polynomial function of H . Let the degree of the polynomial be n , then we write

$$\begin{aligned} \mathcal{E}_n &= \sum_{k=0}^n a_{n+1-k} H^k, \quad n = 1, 2, \dots \\ &= a_1 H^n + a_2 H^{n-1} + \dots + a_n H + a_{n+1}, \end{aligned} \quad (14)$$

where a_k , ($k = 1, 2, \dots$) are constants to be determined. Assuming that the torus is a critical point of the functional \mathcal{F} we shall determine the coefficients

a_i of the polynomial expansion of \mathcal{E} in terms of the torus radii a and r . We shall give three examples here in this section and three more in Appendix A.

1. First Order Functional: $\mathcal{E}_1 = a_1 H + a_2$

Solution:

$$p = -\frac{a_1}{r^2}, \quad a_2 = -\frac{a_1}{r} \quad (15)$$

There is no restriction on the radii a and r . Torus is a critical point of the functional (6) for all values of r and a .

2. Second Order Functional: Most General case: $\mathcal{E}_2 = a_1 H^2 + a_2 H + a_3 + a_4 K$. Here the term containing a_4 is a topologically invariant and gives no contribution to shape equation (6).

Solution:

$$p = -\frac{a_2}{r^2}, \quad a_3 = -\frac{a_2}{r}, \quad a^2 = 2r^2 \quad (16)$$

We find that (for $p = 0$)

$$\mathcal{F}_2 = \int_T \mathcal{E}_2 dA = 2\pi^2 a_1 \quad (17)$$

2-torus with $a^2 = 2r^2$ minimizes the functional $\int_S \mathcal{E}_2 dS$ and the minimum energy is given in (17). Willmore conjecture [4],[22] states that ($a_1 \neq 0$)

$$\frac{1}{a_1} \int_S \mathcal{E}_2 dA \geq 2\pi^2 \quad (18)$$

for all compact surfaces S with genus $g > 0$. The proof of this conjecture has been given very recently [23]. In terms of the Helfrich's functional (1) we have $a_1 = k_c$, $a_2 = 2k_c c_0$ and $a = \frac{1}{2} k_c c_0^2 + w$. The parameters p, w, c_0 and p must satisfy

$$p = \frac{2k_c c_0}{r^2}, \quad w = p r \left(1 + \frac{1}{4} r c_0\right) \quad (19)$$

Since K is a topological invariant and total curvature for torus is zero, there will be no contribution of single K terms in \mathcal{E} . Because of this reason will not include a linear K term in the \mathcal{E}_n expressions.

3. Third Order Functional: $\mathcal{E}_3 = a_1 H^3 + a_2 H^2 + a_3 H + a_4$.

Solution:

$$p = -\frac{3a_1 - a_3 r^2}{r^4}, \quad a_4 = \frac{2a_1 - a_3 r^2}{r^3}, \quad a_2 = \frac{15a_1}{2r}, \quad (20)$$

$$a^2 = (6/5)r^2 \quad (21)$$

We also find that (for $p = 0$)

$$\mathcal{F}_3 = \int_T \mathcal{E}_3 dA = 9\sqrt{5} \pi^2 (a_1/r) \quad (22)$$

2-torus with $a^2 = (6/5)r^2$ minimizes the functional $\int_S \mathcal{E}_3 dS$ and the minimum energy is given in (22). We expect that ($a_1 \neq 0$)

$$(a_1/r)^{-1} \int_S \mathcal{E}_3 dA \geq 9\sqrt{5} \pi^2 \quad (23)$$

for all compact surfaces S with genus $g > 0$. This is the Willmore conjecture for $n = 3$.

It is possible to continue on this study and find infinitely many functionals with Lagrange functions \mathcal{E}_n which are polynomials of degree n . We added three more examples of \mathcal{E}_n with $n = 4, 5, 6$ in Appendix A. We observe that whatever the polynomial degree of the \mathcal{E}_n is, there exists a constraint on the radii a and r of the torus which depend on the degree of the polynomials

$$\frac{a^2}{r^2} = \frac{n^2 - n}{n^2 - n - 1}, \quad n = 2, 3, 4, \dots \quad (24)$$

Hence for all these surfaces $1 < \frac{a^2}{r^2} \leq 2$. The proof of this constraint equation is given in section 5.

These constraints can be utilized to select the correct functional for the toroidal fluid membranes. These functionals are used to minimize the energy of the lipid membranes. The ratio a/r of toroidal configuration can be measured experimentally. Comparing the measured value of this ratio with those listed above (24) we can find the degree of the polynomial function \mathcal{E}_n from (24), hence identifying the functional for the corresponding closed

membrane. As an example, for vesicle membranes such a measurement had been done by Mutz and Bensimon [18]. They measured the value of this ratio approximately as $\frac{a}{r} = 1.43$, or $\frac{a^2}{r^2} = 2.04$. Hence for vesicle membranes the correct functional should be the quadratic one which was first introduced by W. Helfrich several years ago [9]. For other closed fluid membranes the functionals might be different. In his study of vesicles with toroidal topology Seifert [19]-[21] claimed that circular toroidal configurations exist for any $0 < v < 1$ where v is the reduced volume parameter defined by

$$v = \frac{V}{\frac{4\pi}{3}(\frac{A}{4\pi})^{3/2}} \quad (25)$$

Here V is the volume and A is the surface area of the toroidal membranes. In terms of the 2-torus radii we have

$$\frac{a^2}{r^2} = \frac{1}{1.94v^4} \quad (26)$$

Seifert claims also that the membranes with $v \geq 0.84$ are not stable. This corresponds to $\frac{a^2}{r^2} \leq 1.188$. This stability analysis seems to save the quadratic and cubic functionals we have given above. One must be careful about discarding higher order functionals, because the stability analysis mentioned by Seifert is done by using the quadratic functional. To investigate the stability of toroidal configurations with $\frac{a^2}{r^2} = \frac{n^2-n}{n^2-n-1}$, $n = 2, 3, 4, \dots$ one must use the corresponding functional \mathcal{F}_n , ($n = 3, 4, 5, 6, \dots$).

4 Functionals with Mean and Gaussian Curvatures

In this section we shall consider the functionals which are composed of both curvatures K and H . Our main motivation in this section is to look for the possible functionals to remove the conditions (24) on the radii a and r that were obtained by functionals depending only on the mean curvature. We shall show that there are some examples of such functionals where the torus is a critical point without any constraint on radii a and r . Consider now the following functionals \mathcal{F}_n defined by

$$\mathcal{F}_n^K = \int_S \mathcal{E}_n^K(H, K) dA + p \int_V dV \quad (27)$$

where \mathcal{E}_n^K are some functions of H and K for all $(n = 1, 2, 3, \dots)$, and p is a constant and V is the volume enclosed within the surface S . Below a_i , $i = 1 - 7$ are constants. We shall give two examples here and one more example in Appendix B.

1. Third Order: $\mathcal{E}_3^K = a_1 H^3 + a_2 H^2 + a_3 H + a_4 + a_5 K^2 + a_6 H K$

Solution:

$$p = \frac{21a_1 - 2a_2r - 2a_3r^2}{2r^4}, \quad a_6 = \frac{15a_1 - 2a_2r}{2}, \quad (28)$$

$$a_5 = \frac{r(-15a_1 + 2a_2r)}{8}, \quad a_4 = \frac{61a_1 - 6a_2r - 8a_3r^2}{8r^3} \quad (29)$$

with $a^2 = (6/5)r^2$. This is an example where the constraint $a^2 = (6/5)r^2$ at the third order is preserved with the inclusion of K terms into the Lagrange function \mathcal{E}_3 . Adding also H^2K term to \mathcal{E}_3 does not change this fact.

2. Fourth Order: The most general fourth order functional is:
 $\mathcal{E}_4^K = a_1 H^4 + a_2 H^3 + a_3 H^2 + a_4 H + a_5 + a_6 K^2 + a_7 H K + a_8 H^2 K$

Solution:

$$p = \frac{1}{8r^5 \Delta_4} [(-17a^6 + 71a^4r^2 + 130a^2r^4 - 120r^6)a_1 - 8\Delta_4r^2(a_3 + ra_4)], \quad (30)$$

$$a_8 = \frac{a_1}{4\Delta_4} (-55a^6 + 23a^4r^2 - 32a^2r^4 + 144r^6), \quad (31)$$

$$a_7 = \frac{1}{8r\Delta_4} [(5a^6 + 53a^4r^2 + 54a^2r^4 - 72r^6)a_1 - 8\Delta_4r^2a_3], \quad (32)$$

$$a_6 = \frac{1}{32\Delta_4} [(270r^6 - 1233a^4r^2 + 1546r^4a^2 - 648r^6)a_1 + 8r^2\Delta_4a_3], \quad (33)$$

$$a_5 = \frac{1}{32r^4\Delta_4} [(-42a^6 + 193a^4r^2 + 39a^2r^4 - 36r^6)a_1 - 24r^2\Delta_4a_3 - 32r^3\Delta_4a_4], \quad (34)$$

$$a_2 = \frac{a_1}{4r\Delta_4} [-2a^6 - 9a^4r^2 + 38a^2r^4 - 24r^6] \quad (35)$$

where

$$\Delta_4 = (a^2 - 2r^2)(a^2 - r^2)(5a^2 - 6r^2) \quad (36)$$

This example proves that as long as $\Delta_5 \neq 0$ the radii a and r of the 2-torus are left arbitrary for functional \mathcal{F}_4^K with \mathcal{E}_4^K given above. If $\Delta_4 = 0$, since $a > r$, we have two choices $a^2 = 2r^2$ and $a^2 = (6/5)r^2$. Both of these choices lead to $a_1 = 0$ which reduces the degree of the polynomial \mathcal{E}_4 to 3 which is not acceptable.

A special case where $a_8 = 0$:

$$\mathcal{E}_4^K = a_1 H^4 + a_2 H^3 + a_3 H^2 + a_4 H + a_5 + a_6 K^2 + a_7 H K$$

Solution:

$$p = \frac{1168a_1 - 5a_3r^2 - 5a_4r^3}{5r^5}, \quad a_7 = \frac{783a_1 - 5a_3r^2}{5r}, \quad a_2 = \frac{23a_1}{r}, \quad (37)$$

$$a_6 = \frac{r(-783a_1 + 5a_3r^2)}{20}, \quad a_5 = \frac{3369a_1 - 15a_3r^2 - 20a_4r^3}{20r^4} \quad (38)$$

with $a^2 = (12/11)r^2$. This is another example where the constraint (48) is preserved with the inclusion of some nontrivial K terms. We give one more example ($n = 5$) in Appendix B.

5 Main Theorems

We start with the first theorem on the constraint equation in general.

Theorem 1: *2-torus T in \mathbb{R}^3 is a critical point of functionals $\mathcal{F}_n = \int_S \mathcal{E}_n dA + p \int_V dV$ where \mathcal{E}_n are the n th degree polynomials of the mean curvature H of the surface S if and only if $\frac{a^2}{r^2} = \frac{n^2-n}{n^2-n-1}$ for all $n \geq 2$.*

Proof: Using Eqs.(11)-(13) for the 2-torus we obtain

$$\begin{aligned} \nabla^2 H &= \frac{1}{a^2 r^3} [4r^3(-a^2 + r^2)H^3 + 2r^2(5a^2 - 6r^2)H^2 + 4r(-2a^2 + 3r^2)H \\ &\quad + 2(a^2 - 2r^2)], \end{aligned} \quad (39)$$

$$\begin{aligned} (\nabla H) \cdot (\nabla H) &= \frac{1}{a^2 r^4} [4r^4(-a^2 + r^2)H^4 + 4r^3(3a^2 - 4r^2)H^3 + r^2(-13a^2 + 24r^2)H^2 \\ &\quad + 2r(3a^2 - 8r^2)H + 4r^2] \end{aligned} \quad (40)$$

Using these equations for the 2-torus for all $n \geq 2$ we get

$$\nabla^2 H^n = \frac{4n^2}{a^2}(-a^2 + r^2)H^{n+2} + \frac{2}{a^2 r}[(6n^2 - n)a^2 - (8n^2 - 2n)r^2]H^{n+1} + \dots \quad (41)$$

Hence inserting

$$\mathcal{E}_n = a_1 H^n + a_2 H^{n-1} + \dots + a_n H + a_{n+1}, \quad (42)$$

into the general shape equation (6) we get

$$(na_1 \nabla^2 H^{n-1} + (n-1)a_2 \nabla^2 H^{n-2} + \dots) + (4(n-1)a_1 H^{n+1} + 4(n-2)a_2 H^n + \dots) - \frac{2}{r^2}(2rH - 1)(na_1 H^{n-1} + (n-1)a_2 H^{n-2} + \dots) = 0.$$

Using the identity (41) for the 2-torus and collecting the coefficients of the powers of H we get equations for a_i 's. The coefficient of the highest power H^{n+1} in (5) can be calculated exactly

$$a_1 \left[\frac{4n(n-1)^2(-a^2 + r^2)}{a^2} + 4(n-1) \right] H^{n+1} + \dots = 0. \quad (43)$$

Then coefficient of H^{n+1} must vanish which leads to the constraint equations

$$\frac{a^2}{r^2} = \frac{n^2 - n}{n^2 - n - 1},$$

for all $n \geq 2$. The remaining $n+1$ number of equations are linear algebraic equations for a_i , ($i = 1, 2, \dots, n+1$) and p . In general one can solve them in terms of one arbitrary parameter, for instance a_1 . In examples 1-6 in section 3 and Appendix A, the solutions contain two arbitrary coefficients. This means that one of the remaining equations in (43) vanishes identically. This completes the proof of Theorem 1.

Our second theorem is on removing the constraint equations on the radii of the 2-torus by including K terms into the Lagrange functions \mathcal{E}_n .

Theorem 2: *2-torus T in \mathbb{R}^3 is a critical point of functionals $\mathcal{F}_n^K = \int_S \mathcal{E}_n^K dA + p \int_V dV$ where \mathcal{E}_n^K are appropriate polynomials of the mean and Gaussian curvatures (H, K) without any restriction on the radii a and r*

Proof: We modify the Lagrange function \mathcal{E}_n as

$$\mathcal{E}_n^K = \mathcal{E}_n + \sum_{m=1}^{[n/2]} K^m \sum_{k=1}^{n-2m} a_{n+k+1} H^k \quad (44)$$

where $[n/2]$ is $\frac{n}{2}$ for even n and $\frac{n-1}{2}$ for odd n . There are additional $\frac{n^2}{4}$ parameters a_i for even n and $\frac{n^2-1}{4}$ for odd n . The general shape equation (6) will pick up terms from the K dependence hence we also need the following identities for 2-torus

$$\begin{aligned} \nabla \cdot \bar{\nabla} H &= \frac{1}{a^2 r^4} [16r^4(-a^2 + r^2) H^4 + 12r^3(4a^2 - 5r^2) H^3 \\ &\quad + 4r^2(-13a^2 + 21r^2) H^2 + 4r(6a^2 - 13r^2) H \\ &\quad + 4(-a^2 + 3r^2)], \end{aligned} \quad (45)$$

$$\begin{aligned} \nabla \cdot \bar{\nabla} K &= \frac{1}{a^2 r^5} [32r^4(-a^2 + r^2) H^4 + 24r^3(4a^2 - 5r^2) H^3 \\ &\quad + 8r^2(-13a^2 + 21r^2) H^2 + 8r(6a^2 - 13r^2) H \\ &\quad + 8(-a^2 + 3r^2)] \end{aligned} \quad (46)$$

There will be $n + 1 + \frac{n^2}{4}$ parameters a_i in total for even n and $n + 1 + \frac{n^2-1}{4}$ for odd n . On the other the shape equation (6) leads to $n + 1$ number of equations for coefficients a_i and hence the coefficients of H^k terms for $(k = 0, 1, 2, \dots, n, n + 1)$ in (43) are modified and hence the constraint equation disappears. Since number equations are much less than the number of unknowns, modified Eq.(43) will be solved for a_i leaving $\frac{n^2}{4}$ for even n and $\frac{n^2-1}{4}$ for odd n arbitrary. Since the coefficient of H^{n+1} in (43) is modified with the new coefficients then constraint equation disappears. Hence there remains a system of $n + 1$ coupled linear equations for $n + 1$ of coefficients a_i . Provided that determinant Δ_n of the coefficient matrix is different than zero, then solutions of a_i exists in terms of the arbitrary coefficients. If $\Delta_n = 0$ the solution exists but with the constraint equation. This completes the proof of Theorem2. In section 4 and Appendix B we give two examples $n = 4$ and $n = 5$ for illustration of this theorem.

6 Conclusion

Clifford torus is a critical point of Willmore and also Helfrich's functional where the Lagrange function is a quadratic polynomial in the mean curvature of a closed surface in \mathbb{R}^3 . The main contribution of this work is that the sequence of tori $\{T_n\}$ where radii a and r restricted to satisfy $\frac{a^2}{r^2} = \frac{n^2-n}{n^2-n-1}$ for all $n \geq 2$ are the critical points of the functionals \mathcal{F}_n where the Lagrange function \mathcal{E}_n are polynomial of degree n in the mean curvature H of the surface. We have given 6 examples $n = 1 - 6$ and proved this assertion in section 5. If the Lagrange functions \mathcal{E}_n are modified by adding some K terms, Gaussian curvature of the surfaces, it is possible to remove the constraint equations. We have given examples for $n = 4$ and $n = 5$. We have proved in general that 2-torus is a critical point of sequence of functionals \mathcal{F}_n where the Lagrange functions \mathcal{E}_n is a polynomial of both H and K .

In section 3 and Appendix A, for each solution with $n = 2 - 6$ and $p = 0$ we have calculated the curvature energy \mathcal{F}_n . As in the case of the Willmore energy functional ($n = 2$) it is expected that the tori for $n \geq 3$ with the constraints, are minimal energy surfaces. To support this assertion, second variation of the functionals on these surfaces must be calculated. Another point to be examined is the stability of these minimal energy surfaces. These points will be communicated in a forthcoming publication.

7 Appendix A

In this Appendix we add three more examples to those given in section 3.

4. **Fourth Order Functional:** $\mathcal{E}_4 = a_1 H^4 + a_2 H^3 + a_3 H^2 + a_4 H + a_5$.

Solution:

$$p = \frac{77a_1 - a_4r^3}{r^3}, \quad a_5 = \frac{51a_1 - a_4r^3}{r^4}, \quad a_3 = \frac{783a_1}{5r^2}, \quad a_2 = \frac{23a_1}{r}, \quad (47)$$

$$a^2 = (12/11)r^2 \quad (48)$$

We find that (for $p = 0$)

$$\mathcal{F}_4 = \int_T \mathcal{E}_4 dA = \frac{666\sqrt{11}\pi^2}{5} \left(\frac{a1}{r^2}\right) \quad (49)$$

5. **Fifth Order Functional:** $\mathcal{E}_5 = a_1 H^5 + a_2 H^4 + a_3 H^3 + a_4 H^2 + a_5 H + a_6$.

Solution:

$$p = \frac{41915 a_1 - 14 a_5 r^4}{14 r^6}, \quad a_6 = \frac{13848 a_1 - 7 a_5 r^4}{7 r^5}, \quad (50)$$

$$a_4 = \frac{715045 a_1}{126 r^3}, \quad a_3 = \frac{12035 a_1}{14 r^2}, \quad a_2 = \frac{50 a_1}{r}, \quad (51)$$

$$a^2 = (20/19) r^2 \quad (52)$$

we find that (for $p = 0$)

$$\mathcal{F}_5 = \int_T \mathcal{E}_5 dA = \frac{235750 \sqrt{19}}{63} \pi^2 (a_1 / r^3) \quad (53)$$

6. **Sixth Order Functional:** $\mathcal{E}_6 = a_1 H^6 + a_2 H^5 + a_3 H^4 + a_4 H^3 + a_5 H^2 + a_6 H + a_7$.

Solution:

$$p = \frac{544813 a_1 - 32 a_6 r^5}{32 r^7}, \quad a_7 = \frac{1796815 a_1 - 16 a_6 r^5}{16 r^6}, \quad (54)$$

$$a_5 = \frac{139780065 a_1}{448 r^4}, \quad a_4 = \frac{1534015 a_1}{32 r^3}, \quad (55)$$

$$a_3 = \frac{6235 a_1}{2 r^2}, \quad a_2 = \frac{183 a_1}{2 r}, \quad (56)$$

$$a^2 = (30/29) r^2 \quad (57)$$

we find that (for $p = 0$)

$$\mathcal{F}_6 = \int_T \mathcal{E}_6 dA = \frac{37643625 \sqrt{29}}{224} \pi^2 (a_1 / r^4) \quad (58)$$

8 Appendix B

In this Appendix we add one more example those given in section 4.

3. **Fifth Order:** The most general fifth order functional is:

$$\mathcal{E}_4^K = a_1 H^5 + a_2 H^4 + a_3 H^3 + a_4 H^2 + a_5 H + a_6 + a_7 K^2 H + a_8 K^2 + a_9 K H^3 + a_{10} K H^2 + a_{11} K H$$

Solution:

$$p = \frac{1}{96r^6 \Delta_5} [(-150a^8 + 1174a^6r^2 + 7218a^2r^6 - 2520r^8)a_1 + r(-240a^8 + 930a^4r^2 + 1554a^4r^4 - 4116a^2r^7 + 1872r^8)a_2 - 72r^2\Delta_5 a_3 - 480r^3\Delta_5 a_4 - 96r^4\Delta_5 a_5], \quad (59)$$

$$a_{11} = \frac{1}{48r^2\Delta_5} [(66a^8 - 721a^6r^2 - 4a^4r^4 + 924a^2r^6 - 480r^8)a_1 + r(18a^8 + 246a^6r^2 + 288a^4r^4 + 1128a^2r^6 + 576r^8)a_2 - 24r^2\Delta_5(a_3 + 2ra_4)], \quad (60)$$

$$a_{10} = \frac{1}{24r} [(226a^8 - 1141a^6r^2 + 2161a^4r^4 - 1866a^2r^6 + 600r^8)a_1 + r(-342a^8 + 1704a^6r^2 - 3700a^4r^4 + 2436a^2r^4 - 720r^8)a_2 - 24r^2\Delta_5 a_3], \quad (61)$$

$$a_9 = \frac{1}{6\Delta_5} (a^2 - r^2)(a^2 - 2r^2)(a^2 - \frac{20}{19}r^2)(95a^2 - 114r^2), \quad (62)$$

$$a_8 = \frac{1}{192r\Delta_5} (-2702a^8 + 11838a^6r^2 - 25525a^4r^4 + 18894a^2r^6 - 5880r^8)a_1 + r(1644r^8 - 8934a^6r^2 + 16230a^4r^4 - 12540a^2r^6 + 3600r^8)a_2 + 48r^3\Delta_5(a_3 + ra_4), \quad (63)$$

$$a_7 = \frac{1}{32\Delta_5} [(358a^8 - 1948a^6r^2 + 3833a^4r^4 - 3238a^2r^6 + 1000r^8)a_1 + r(4a^8 + 14a^6r^2 - 94a^4r^4 + 124a^2r^6 - 48r^8)a_2 + 8\Delta_5r^2a_3], \quad (64)$$

$$a_6 = \frac{1}{64r^5\Delta_5} [(-60a^8 + 458a^6r^2 - 2913r^4a^4 + 3398a^2r^6 - 1208r^8)a_1 + r(-100a^8 + 418a^6r^2 + 766a^4r^4 - 1996a^2r^6 + 912r^8)a_2 - 32r^2\Delta_5a_3 - 48r^3\Delta_5a_4 - 64r^4\Delta_5a_5], \quad (65)$$

where

$$\Delta_5 = (a^2 - r^2)^2 (a^2 - 2r^2)(5a^2 - 6r^2). \quad (66)$$

Again as long as $\Delta_5 \neq 0$ we have here another example of a functional where an arbitrary 2-torus is a critical point. If $\Delta_5 = 0$, since $a > r$, we have only two cases:

Case 1: $a^2 = 2r^2$.

$$p = \frac{1}{108r^6} [-31a_1 - 9a_3r^2 + 36a_5r^4 + 144a_6r^5], \quad (67)$$

$$a_{11} = \frac{1}{54r^2} [97a_1 + 9a_3r^2 + 72a_5r^4 + 72a_6r^5], \quad (68)$$

$$a_{10} = -\frac{1}{9r} [38a_1 + 9a_3r^2], \quad a_9 = -3a_1, \quad (69)$$

$$a_8 = \frac{1}{108r} [163a_1 + 9a_3r^2 - 36a_5r^4 - 36a_6r^5], \quad (70)$$

$$a_7 = \frac{1}{36} [73a_1 + 9a_3r^2], \quad a_2 = \frac{35a_1}{18r}. \quad (71)$$

The coefficients a_1, a_3, a_4, a_5 and a_6 are left arbitrary.

Case 2: $a^2 = \frac{6}{5}r^2$.

$$p = \frac{1}{8424r^6} [-9119a_1 - 486a_3r^2 + 324a_4r^3 + 3240a_5r^4 + 11664a_6r^5], \quad (72)$$

$$a_{11} = \frac{1}{21060r^2} [88723a_1 - 162a_3r^2 + -5508a_4r^3 + 20736a_5r^4 + 20736a_6r^5], \quad (73)$$

$$a_{10} = -\frac{1}{10530r} [-155972a_1 - 9882a_3r^2 + 972a_4r^3 + 296a_5r^4 + 1296a_6r^5], \quad (74)$$

$$a_8 = \frac{1}{42120r} [377387a_1 + 5022a_3r^2 + 2268a_4r^3 - 11016a_5r^4 - 11016a_6r^5], \quad (75)$$

$$a_7 = \frac{1}{42120} [39317a_1 + 9882a_3r^2 - 912a_4r^3 - 1296a_5r^4 - 1296a_6r^5], \quad (76)$$

$$a_9 = -\frac{7a_1}{5}, \quad a_2 = \frac{791a_1}{162r}. \quad (77)$$

The coefficients a_1, a_3, a_5 and a_6 are left arbitrary.

We claim that for all functionals with $\mathcal{E}_n = a_1 H^n + \dots$ discussed in section 3, it is possible to add suitable K terms so that an arbitrary 2-torus is a critical point without any constraint on the radii a and r . We prove our claim in section 5.

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References

- [1] R. Parthasarathy and K. S. Viswanathan, *Geometric properties of QCD string from Willmore functional*, J. Geom. Phys. **38**, 207-216 (2001) .
- [2] L.P. Eisenhart, *A Treatise on the Differential Geometry of Curves and Surfaces*, Dover Pub., Inc., New York, (1909).
- [3] M. P. Do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice-Hall, Inc., Englewood Cliffs, NJ, (1976).
- [4] T.J. Willmore, *Total Curvature in Riemannian Geometry*, John Willey and Sons, New York, (1982).
- [5] T.J. Willmore, *Surfaces in Conformal Geometry*, Annals of Global Analysis and Geometry **18**, 255-264 (2000).
- [6] A. Sym, *Soliton Surfaces*, Lett. Nuovo Cimento **33**, 394-400 (1982).
- [7] A.S. Fokas and I.M. Gelfand, *Surfaces on Lie Groups, on Lie Algebras, and Their Integrability*, Commun. Math. Phys. **177**, 203-220 (1996).
- [8] M. Gürses and S. Tek, *Koteweg de Vries Surfaces*, Nonlinear Analysis: Theory, Methods and Applications **95**, 11-22 (2014). (See also the references therein)
- [9] W. Helfrich, *Elastic Properties of lipid bilayers: theory and possible experiments*, Z. Naturforsch. **28**, 693 (1973).
- [10] Z.C. Tu and Z.C. Ou-Yang, *A Geometric Theory on the Elasticity of Bio-membranes*, J. Phys. A: Math. Gen. **37**, 11407-11429 (2004).
- [11] Z.C. Tu and Z.C. Ou-Yang, *Lipid Membranes with Free Edges*, Phys. Rev. E **68**, 061915 (2003).

- [12] Z. C. Tu and Z. C. Ou-Yang, Proceeding of the Seventh International Conference on Geometry, Integrability and Quantization, Varna, Bulgaria, 2005, edited by I. M. Mladenov and M. de León, SOFTEX, Sofia, 237 (2005), Report No. math-ph/0506055.
- [13] Z. C. Ou-Yang and W. Helfrich, *Instability and Deformation of a Spherical Vesicle by Pressure*, Phys. Rev. Lett. **59**, 2486-2488 (1987).
- [14] Z. C. Ou-Yang and W. Helfrich, *Bending Energy of Vesicle Membranes: General Expansion for the First, Second, and Third Variation of the Shape Energy and Applications to Sphere and Cylinders*, Phys. Rev. A **39**, 5280-5288 (1989).
- [15] Z. C. Ou-Yang, J. Liu and Y. Xie, *Geometric Methods in the Elastic Theory of Membranes in Liquid Crystal Phases*, World Scientific, Singapore, (1999).
- [16] Z.C. Tu, *Elastic Theory of Biomembranes*, Thin Solid Films **393**, 19-23 (2001).
- [17] Qu-Yang Zhong-can, *Anchor ring-vesicle membranes*, Phys. Rev. A **41**, 4517 (1990).
- [18] M. Mutz and D. Bensimon, *Observation of toroidal vesicles*, Phys. Rev. A **43**, 4525 (1991).
- [19] U. Seifert, *Vesicles of Toroidal Topology*, Phys. Rev. Lett. **66**, 2404 (1991).
- [20] U. Seifert, *Configurations of fluid membranes and vesicles*, Advances in Physics. **46**, 13-137 (1997).
- [21] M.I.G. Bloor and Michael J. Wilson, *Method for efficient shape parametrization of fluid membranes and vesicles*, Phys. Rev. **D61**, 4218-4229 (2000).
- [22] T. J. Willmore, *Note on embedded surfaces*. An. Sti. Univ. "Al. I. Cuza" Iasi Sect. I a Mat. (N.S.) 11B,493-496(1965).

- [23] Fernando C. Marques and Andr  Neves, *Min-Max theory and the Willmore conjecture*, Annals of Mathematics, **179.2**, 683-782 (2014): arXiv:1202.6036v2 [math.DG].